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Observations on the dynamics of the two-dimensional vortex gas on compact Riemann surfaces

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Abstract. The dynamics and symmetries of the two-dimensional vortex gas on compact Riemann Surfaces are analysed using Lagrangian dynamics. As the vortex Lagrangian is linear in the canonical momenta, Dirac's theory of constraints is then used to form the Hamiltonian dynamics for the system.

1. Introduction

In our previous paper [1] we have been mostly concerned with demonstrating the existence of vortex states in the excitation spectrum of two-dimensional ^{4}He liquids. We now turn our attention to the analysis of the dynamical properties of the system by treating the vortices as point particles moving on a general compact Riemann Surface. While much of this analysis will be done from the Lagrangian point of view, most of the interest in the two-dimensional vortex gas is due to the Kosterlitz–Thouless phase transition. What is then of applicational importance is not the Lagrangian formulation, but rather the Hamiltonian one, and we shall also be interested in the Hamiltonian description of the vortex dynamics.

We caution the reader that what we are dealing with in this paper are *quantum vortices* which arise from the multiplicity of the phase of microscopic quantum mechanical field ψ for ^{4}He atoms. The vortex Lagrangian will be obtained from the Lagrangian for the non-relativistic bose gas and *not* from any fluid-dynamical arguments. While a very fruitfull analogy may be made between vortices arising from actual current flows in a liquid and vortices arising from the multiplicity of the phase of ψ, this is only an analogy that need not necessarily hold in all cases. In fact, one of the aims of this paper is to present a derivation and analysis of the quantum vortex gas which makes as little use of this analogy as possible. Another aim of this paper is to provide a systematic foundation for the dynamics of the two-dimensional vortex gas so that the statistical-mechanical properties of the system may be analysed. One cannot fully understand the statistical mechanics of a system until one has understood its dynamics. The grand-canonical ensemble cannot even be constructed until all the constants of the systems are known. It is for this reason that we also include a discussion on the Hamiltonian dynamics of the vortex gas, particularly from Dirac's theory of constraints point of view.

The rest of this paper is organized in the following manner. In section 2 we present a derivation of the Lagrangian of the two-dimensional vortex gas on arbitrary Reimann Surfaces. This is a generalization of our previous work [1]. In section 3 we formulate the

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vortex phase space and in section 4 the constants of the motion for the system are analysed. Finally, in section 5, Dirac's theory of constraints is used to formulate the Hamiltonian dynamics of the system and its usefulness in the analysis of the Kosterlitz–Thouless phase transition is made clear.

2. The Vortex Lagrangian

Our starting point will be the functional integral

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp\left(-\mathrm{i}\int \mathcal{L}\mathrm{d}t\right) \tag{2.1}$$

where $\mathcal{L}$ is the standard *microscopic* Lagrangian for a system of non-relativistic ^{4}He atoms constrained to move on a two-dimensional manifold $\mathbf{M}$

$$\mathcal{L} = \int_{\mathbf{M}} \left\{ \frac{\mathrm{i}}{2}\left(\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t}\psi\right) - \frac{1}{2} g^{ab}\partial_a \psi^\dagger \partial_b \psi - F[|\psi|^2]\right\}\sqrt{g}\,\mathrm{d}^2x\,. \tag{2.2}$$

With the replacement $t \longrightarrow \mathrm{i}\tau$, $0 \leqslant \tau \leqslant \beta$, $\mathcal{Z}$ becomes the partition function for the ^{4}He atoms. F is any functional of the density and we have taken $\hbar = m = 1$. The metric g_{ab} on $\mathbf{M}$ is considered to be given and fixed.

The function ψ is, as usual, a complex field representing the possible states of the helium atoms and is characterized mathematically as a section of a non-trivial complex line bundle over $\mathbf{M}$. Although the functional integral in (2.1) is over all such C^∞ sections, standard techniques used in evaluating the integral do not take into account those fields which lie in the kernel of Laplace's operator, namely holomorphic and anti-holomorphic fields. As was shown in [1] it is precisely the integral over these fields which gives rise to the path integral for the two-dimensional vortex gas. For this reason we will usually take ψ to be holomorphic in the rest of the paper.

As we shall see, the vortex Lagrangian will come solely from the free Lagrangian

$$\mathcal{L}_l \equiv \int_{\mathbf{M}} \left\{ \frac{\mathrm{i}}{2}\left(\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{\partial \psi^\dagger}{\partial t}\psi\right) - \frac{1}{2} g^{ab}\partial_a \psi^\dagger \partial_b \psi \right\}\sqrt{g}\,\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{2.3}$$

and is independent of any specific choice for F. We start by focusing our attention on the kinetic energy $\mathcal{K}$ part of $\mathcal{L}_l$. Since F does not contain any gradient terms, the conserved current is still the 1-form $j = \mathrm{i}(\mathrm{d}\psi^\dagger\,\psi - \psi^\dagger \mathrm{d}\psi)/2$. The conservation equation, however, is now $\dot{n} = \delta j$ where $n \equiv |\psi|^2$ and δ is the co-exterior derivative. Following Dashen and Sharp [2], we write $\mathcal{K}$ as

$$\mathcal{K} = \frac{1}{8}\int_{\mathbf{M}} n^{-1}\delta\rho \wedge *\delta\rho + \frac{1}{2}\int_{\mathbf{M}} n^{-1} j \wedge *j \tag{2.4}$$

where $\rho = n\sqrt{g}, \mathrm{d}x^1 \wedge, \mathrm{d}x^2$.

Choose a local coordinate system (x^1, x^2) on a neighborhood of $\mathbf{M}$. Since $\mathbf{M}$ is a Riemann Surface, its metric is simply $\mathrm{d}s^2 = 2g_{z\bar{z}}\mathrm{d}z \otimes \mathrm{d}\bar{z}$ where $z \equiv (x^1 + \mathrm{i}x^2)/\sqrt{2}$. Then $\sqrt{g} = 2g_{z\bar{z}}$ and $\mathcal{K}$ becomes

$$\mathcal{K} = \int_{\mathbf{M}} \left(\tfrac{1}{4} n^{-1}\partial_z n \partial_{\bar{z}} n + n^{-1} j_z j_{\bar{z}}\right)\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{2.5}$$

where $j_z \equiv (j_1 - \mathrm{i} j_2)/\sqrt{2}$. This is exactly the same as the free Hamiltonian for helium atoms on the plane. No reference to the metric of the surface is present. Moreover, for ψ holomorphic, $j = -\delta\rho/2 \Longrightarrow \dot{n} = 0$ and $\mathcal{K}$ has a remarkably simple form

$$\mathcal{K} = \tfrac{1}{4} \int_{\mathbf{M}} n \delta\eta \wedge *\delta\eta \tag{2.6}$$

where $\eta \equiv \log n\sqrt{g}\, \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$.

The Green's function for Laplace's operator is introduce into (2.6) by using the Hodge decomposition theorem [3] which states that any p-form α may be decomposed into the sum of three pieces:

$$\alpha = \mathrm{d}G[\delta\alpha] + \delta G[\mathrm{d}\alpha] + H[\alpha]. \tag{2.7}$$

H is a projection operator which maps α into the space of harmonic p-forms. As we are dealing with the physical observables n and j, it suffices to consider the space of *global* harmonic forms. Since the only harmonic form on a compact manifold without boundary is the constant form, the components of $H[\alpha]$ are simply the spatial average of the components of α over $\mathbf{M}$.

G is Green's operator and its action on forms is as follows. We define the delta function on the surface as

$$f(z) = \int_{\mathbf{M}} f(w)\delta^2(z-w)\, \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{2.8}$$

for any function f. For the 2-form $\beta = b(z)\sqrt{g}\, \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$

$$G[\beta] \equiv \left(\int_{\mathbf{M}} \{b(w) - b_o\}\phi(z-w)\sqrt{g(w)}\, \mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} \right) \sqrt{g}\, \mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{2.9}$$

where b_0 is the spatial average of b over the surface and $\phi(z-w) \equiv -\log|z-w|^2/4\pi$ satisfies

$$-2\partial_z \partial_{\bar{z}} \phi(z-w) = \delta^2(z-w). \tag{2.10}$$

For the 1-form $c = c_z \mathrm{d}z + c_{\bar{z}} \mathrm{d}\bar{z}$

$$G[c] \equiv \left(\int_{\mathbf{M}} \{c_w(w) - c_{wo}\}\phi_z(z-w)\, \mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} \right) \mathrm{d}z + \left(\int_{\mathbf{M}} \{c_{\bar{w}}(w) - c_{\bar{w}o}\}\phi_{\bar{z}}(z-w)\, \mathrm{i}\, \mathrm{d}w \wedge \mathrm{d}\bar{w} \right) \mathrm{d}\bar{z} \tag{2.11}$$

where once again c_{z0} is the spatial average of c_z. The function $\phi_z(z-w)$ is the solution of the partial differential equation

$$-2\partial_z \left(\sqrt{g}^{-1} \partial_{\bar{z}} \phi_z \right) = \delta^2(z-w) \tag{2.12}$$

and is related to $\phi(z-w)$ through the integral equation

$$\phi_z(z-w) = 2 \int_{\mathbf{M}} \partial_{\bar{u}} \phi(u-w) \sqrt{g(u)}\, \partial_u \phi(u-z)\, \mathrm{i}\, \mathrm{d}u \wedge \mathrm{d}\bar{u} \tag{2.13}$$

as can be readily verified.

The density is now decomposed into

$$n = n_0 + \frac{1}{4\pi}\int_M \left(\frac{\partial_w n}{\bar{z}-\bar{w}} + \frac{\partial_{\bar{w}} n}{z-w}\right) \mathrm{i}\,\mathrm{d}w \wedge \mathrm{d}\bar{w} \tag{2.14}$$

while $\delta\eta$ becomes

$$\delta\eta = \mathrm{i}\left(\frac{1}{2\pi}\int_M \frac{\partial_w\partial_{\bar{w}}\log n}{z-w}\,\mathrm{i}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}\right)\mathrm{d}z - \mathrm{i}\left(\frac{1}{2\pi}\int_M \frac{\partial_w\partial_{\bar{w}}\log n}{\bar{z}-\bar{w}}\,\mathrm{i}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}\right)\mathrm{d}\bar{z}. \tag{2.15}$$

$H[\delta\eta]$ vanishes by Stoke's theorem. Putting (2.14) and (2.15) into (2.6), we find that all the integrals over z become trivial and can be reduced into doing the single integral

$$\int_M \frac{1}{(w-z)(\bar{u}-\bar{z})}\,\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = -2\pi\log|w-u|^2. \tag{2.16}$$

Then returning to the free Lagrangian $\mathcal{L}_0$, we obtain

$$\begin{aligned}\mathcal{L}_0 = \int_M \mathrm{i}\frac{n}{2}\frac{\partial}{\partial t}\log\left(\frac{\psi}{\psi^\dagger}\right)\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &+ \frac{1}{4\pi}\iint n(w)\partial_w\partial_{\bar{w}}\log n\;\partial_u\partial_{\bar{u}}\log n\;\log|w-u|^2\;\mathrm{i}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}\;\mathrm{i}\,\mathrm{d}u\wedge\mathrm{d}\bar{u} \\ &- \frac{1}{4\pi}\iint n^{-1}\partial_w n\,\partial_{\bar{w}}n\;\partial_u\partial_{\bar{u}}\log n\;\log|w-u|^2\,\mathrm{i}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}\;\mathrm{i}\,\mathrm{d}u\wedge\mathrm{d}\bar{u}.\end{aligned} \tag{2.17}$$

At this point we make use of the following lemma from Gunning [4].

Lemma. Let $\xi_{\alpha\beta}$ be the transition functions of a line bundle ξ subordinate to an open covering $\{U_\alpha\}$ of the compact manifold **M**. Suppose that $\{r_\alpha\}$ are nowhere vanishing C^∞ functions defined on U_α satisfying

$$r_\alpha(p) = |\xi_{\alpha\beta}|^2 r_\beta(p) \tag{2.18}$$

for a point $p \in U_\alpha \cap U_\beta$. Then $\partial_z\partial_{\bar{z}}\log r_\alpha \mathrm{d}z\wedge\mathrm{d}\bar{z}$ is a well defined 2-form on **M** and the chern class $c(\xi)$ of the line bundle is

$$c(\xi) = \frac{1}{2\pi\mathrm{i}}\int \partial_z\partial_{\bar{z}}\log r\,\mathrm{d}z\wedge\mathrm{d}\bar{z}. \tag{2.19}$$

Noticing that the density transforms in exactly the same way as r, we make the following identifications:

$$\partial_z\partial_{\bar{z}}\log n \Longleftrightarrow 2\pi\sum_{\alpha=1}^{N} q_\alpha\delta^2(z-z^\alpha) \tag{2.20}$$

and

$$\log\left(\frac{\psi}{\psi^\dagger}\right) \Longleftrightarrow \sum_{\alpha=1}^{N} q_\alpha\log\left(\frac{z-z^\alpha}{\bar{z}-\bar{z}^\alpha}\right) \tag{2.21}$$

where z^α is the *location* of the vortex with *circulation* q_α and N is the total number of vortices. It should be stressed, however, that these identifications are for convenience only. We are not requiring that either the density or the phase of ψ have these properties. They are instead a signature of the non-triviality of the complex line bundle. Notice moreover that the lemma, and thus the identifications, hold even when n is a constant.

Using these identifications in (2.17), and labelling the Lagrangian for the vortex gas as $\mathcal{L}_v$ to avoid confusing it with the free Lagrangian for the ^{4}He atoms, we obtain

$$\mathcal{L}_v = \frac{i}{2}\sum_{\alpha=1}^{N} q_\alpha\left(\frac{dz^\alpha}{dt}\bar{p}_\alpha - \frac{d\bar{z}^\alpha}{dt}p_\alpha\right) - \mathcal{K} \tag{2.22}$$

where

$$\mathcal{K} = -\pi\sum_{\alpha,\beta=1}^{N} n(z^\alpha)q_\alpha q_\beta \log|z^\alpha - z^\beta|^2 + \tfrac{1}{2}\sum_{\alpha}^{N} q_\alpha \int_{\mathbf{M}} n^{-1}\partial_u n \partial_{\bar{u}} n \log|u - z^\alpha|^2 \, i\, du \wedge d\bar{u}\,. \tag{2.23}$$

and

$$\bar{p}_\alpha \equiv \int_{\mathbf{M}} \frac{n\sqrt{g}}{z^\alpha - z}\, i\, dz \wedge d\bar{z}\,. \tag{2.24}$$

From Dolbeault's Lemma [5]

$$\frac{\partial \bar{p}_\alpha}{\partial \bar{z}^\alpha} = 2\pi\left(n\sqrt{g}\right)_\alpha\,. \tag{2.25}$$

By $\left(n\sqrt{g}\right)_\alpha$ we mean the value of $n\sqrt{g}$ evaluated at z^α.

For ψ holomorphic, the microscopic ^{4}He current $j = -\delta\rho/2$ where $\rho \equiv n\sqrt{g}\, i\, dz \wedge d\bar{z}$. From the conservation equation $\dot{n} = \delta j$, $\dot{n} = 0$. This ensures that the interaction Hamiltonian $\mathcal{K}$ and, as we shall see later, the total vortex Hamiltonian $\mathcal{H}_v$, is time-independent and conserved. The vortices themselves form a conserved, dynamical system and there is no exchange of energy between them and the underlying ^{4}He fluid. The fluid density n is treated here as an all-pervasive external field.

We now see that the form of the vortex Lagrangian, and consequently the very peculiar first-order vortex evolution equations, are inherited from a natural choice for the Lagrangian of the microscopic ^{4}He atoms. Like $\mathcal{L}$, $\mathcal{L}_v$ is first-order in the canonical momenta and thus the vortex Hamiltonian $\mathcal{H}$ obtained from $\mathcal{L}_v$ is simply $\mathcal{K}$. The first part of $\mathcal{H}$ is the vortex–vortex interaction Hamiltonian and contains the usual self-interaction piece. The second part is a vortex–quasi-particle interaction Hamiltonian. Notice that when the density is constant, the second part of $\mathcal{H}$ vanishes and we are back to the original Kosterlitz–Thouless coulombic interaction.

A few words should be said about $\mathcal{L}_v$, particularly about the form $\mathcal{K}$ takes. For constant n, $\mathcal{K}$ reduces to the standard logarithmic interaction Hamiltonian for point vortices *on the infinite plane*. It thus has the same form independent of the specific choice of the manifold **M**, which is quite peculiar, and counter-intuitive. Note, however, that $\mathcal{K}$ comes from the free part of the ^{4}He Hamiltonian (2.3) for which, as we have seen, the metric drops out completely (2.5). It is for this reason that no mention of the metric on **M** is left in (2.23). In our analysis we have implicitly mapped **M** onto the Reimann plane. The metric *does*, however, occur in $\mathcal{L}_v$ since the velocity terms in the Lagrangian comes from the $\psi^\dagger\dot{\psi}$ term in (2.3) for which $\sqrt{g}$ is still present. We shall show in section 3 that there are choices of local coordinate systems for which this piece of the Lagrangian may also be mapped onto the Reimann plane. The analysis is somewhat more involved as the structure of the vortex phase space, which is related to the global properties of **M**, is determined from this term.

3. The vortex phase space

Taking $\bar{z}^\alpha$ as our generalized coordinate $\mathbf{q}^\alpha$, its canonical momenta $\mathbf{p}_\alpha$ are

$$\mathbf{p}_\alpha = -\frac{\mathrm{i}}{2}q_\alpha \int_{\mathbf{M}} \frac{n\sqrt{g}}{\bar{z}^\alpha - \bar{z}}\,\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{3.1}$$

and the equations of motion are

$$2\pi q_\alpha (n\sqrt{g})_\alpha \mathrm{i}\frac{\mathrm{d}z^\alpha}{\mathrm{d}t} = \frac{\partial \mathcal{K}}{\partial \bar{z}^\alpha}\,. \tag{3.2}$$

Let $G_{\alpha\bar{\beta}} = q_\alpha (n\sqrt{g})_\alpha \delta_{\alpha\bar{\beta}}/n_0$ where n_0 is the spatial average of n over **M**. As we are dealing with *liquid* ^{4}He, we will require that n be nowhere vanishing on **M**. Since $\sqrt{g}$ is also nowhere vanishing, $G_{\alpha\bar{\beta}}$ is a Hermitian matrix with a well defined inverse for all z^α. It is identified with the metric on the vortex phase space **K**, a Hermitian manifold with complex dimension N.

The 2-form

$$\Phi \equiv \mathrm{i}\sum_{\alpha,\beta=1}^{N} G_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta \tag{3.3}$$

is the second fundamental form on **K**. Since

$$\frac{\partial G_{\alpha\bar{\beta}}}{\partial z^\lambda} = \frac{q_\alpha}{n_0}\frac{\partial (n\sqrt{g})_\alpha}{\partial z^\alpha}\delta_{\alpha\lambda}\delta_{\alpha\bar{\beta}} \tag{3.4}$$

it is also closed and **K** is a Kähler manifold with a natural symplectic structure. In particular, there exists a local coordinate system $\{w^\alpha, \bar{w}^\alpha\}$ on **K** such that [5]

$$\Phi = \mathrm{i}\sum_{\alpha,\beta}^{N} q_\alpha \mathrm{d}w^\alpha \wedge \mathrm{d}\bar{w}^\alpha \tag{3.5}$$

with the corresponding equations of motion

$$2\pi n_0 \mathrm{i} q_\alpha \frac{\mathrm{d}w^\alpha}{\mathrm{d}t} = \frac{\partial \mathcal{K}}{\partial \bar{w}^\alpha}\,. \tag{3.6}$$

These, in the constant density limit, are Kirchoff's equations for the motion of vortices on a plane [6]. For any two differentiable finctions of z^α and $\bar{z}^\alpha$, the Poisson bracket on **K** is then defined as

$$\{f, g\} \equiv \frac{1}{2\pi \mathrm{i} n_0}\sum_{\alpha,\beta=1}^{N} G^{\alpha\bar{\beta}}\left(\frac{\partial f}{\partial z^\alpha}\frac{\partial g}{\partial \bar{z}^\beta} - \frac{\partial g}{\partial z^\beta}\frac{\partial f}{\partial \bar{z}^\alpha}\right). \tag{3.7}$$

Let $\{U_\alpha\}$ be an open covering of $\mathbf{M}$ such that $z^\alpha \in U_\alpha$ but $z^\alpha \notin U_\beta$ for $\alpha \neq \beta$. Define on U_α the local function

$$G|_{U_\alpha} \equiv \frac{q_\alpha}{2\pi} \int_{U_\alpha} \frac{(n\sqrt{g})}{n_0} \log|z - z^\alpha|^2 \,\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{3.8}$$

On this open neighbourhood

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G|_{U_\alpha}}{\partial z^\alpha \partial \bar{z}^\beta}\,. \tag{3.9}$$

The *Kähler potential* G for $\mathbf{K}$ is defined as the formal sum

$$G \equiv \sum_\alpha^N G|_{U_\alpha} \tag{3.10}$$

and is unique up to a holomorphic function in z^α. The momentum canonical to $\mathbf{q}_\alpha$ is related to G by

$$\mathbf{p}_\alpha = -\pi n_0 \mathrm{i} \frac{\partial}{\partial \mathbf{q}^\alpha} G\,. \tag{3.11}$$

One naively expects the phase space to have a simple product structure $\mathbf{K} = \mathbf{M}\times\cdots\times\mathbf{M}$. Due to the presence of n in the metric $G_{\alpha\bar{\beta}}$ this, however, does not turn out to be the case. The fluid density also plays a role in determining the *global* structure of the phase space. Physically, this is interpreted as an indication that the vortices do not move on the surface of $\mathbf{M}$ but rather 'float' on top of the fluid. This effective surface $\mathbf{M}_{\text{eff}}$ that the vortices move on has the metric $\mathrm{d}s^2 = 2ng_{z\bar{z}}\mathrm{d}z \otimes \mathrm{d}\bar{z}$. Notice also that because

$$n_0 \equiv \int_{\mathbf{M}} n\sqrt{g}\ \mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \Big/ \int_{\mathbf{M}} \sqrt{g}\ \mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{3.12}$$

the volume $\mathrm{vol}\,\mathbf{K}$ of the phase space

$$\mathrm{vol}\,\mathbf{K} \equiv \int_{\mathbf{K}} \frac{\Phi^N}{N!} = \left|\prod_\alpha^N q_\alpha\right| (\mathrm{vol}\,\mathbf{M})^N \tag{3.13}$$

can be defined and is finite as long as the volume of $\mathbf{M}$ is. The absolute value of $\prod q_\alpha$ is taken to ensure that a consistent orientation is preserved thoughout the phase space and to ensure that $\mathrm{vol}\,\mathbf{K}$ is positive.

4. Symmetries of $\mathcal{L}_\mathrm{v}$

Let $h_\alpha(z^\alpha)$ be a holomorphic function of z^α only and consider the re-definition

$$\bar{p}'_\alpha = \bar{p}_\alpha + \frac{\partial h_\alpha}{\partial z^\alpha} \tag{4.1}$$

which changes $\mathcal{L}_\mathrm{v}$ by a total derivative

$$\mathcal{L}'_\mathrm{v}[\bar{p}'_\alpha] = \mathcal{L}_\mathrm{v}[\bar{p}_\alpha] + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{i}}{2}\sum_\alpha^N q_\alpha[h_\alpha - \bar{h}_\alpha]\right). \tag{4.2}$$

As $\mathcal{L}_v$ is unique only up to a total derivative, $\bar{p}_\alpha$ is determined only up to a holomorphic function in z^α. We shall call this freedom in choosing h_α a *gauge freedom* and we say that $\bar{p}_\alpha$ is *equivalent* to $\bar{p}'_\alpha$ if they differ by at most a choice of gauge.

Next, consider the open covering $\{U_\alpha\}$ of $\mathbf{M}$ defined in section 4. By definition,

$$\bar{p}_\alpha = \bar{p}_\alpha|_{U_\alpha} + \sum_{\beta\neq\alpha}^{N} \int_{U_\beta} \frac{n\sqrt{g}}{z^\alpha - z}\,\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{4.3}$$

where

$$\bar{p}_\alpha|_{U_\alpha} \equiv \int_{U_\alpha} \frac{n\sqrt{g}}{z^\alpha - z}\,\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{4.4}$$

Since $z^\alpha \notin U_\beta$ for $\alpha \neq \beta$, each term in the sum on the right-hand side of (4.3) is a holomorphic function in z^α. $\bar{p}_\alpha$ is then equivalent to $\bar{p}_\alpha|_{U_\alpha}$. Similarly, given a different open covering $\{V_\alpha\}$ of $\mathbf{M}$ satisfying the same conditions outlined in section 3, $\bar{p}_\alpha|_{V_\alpha}$ is also equivalent to $\bar{p}_\alpha|_{U_\alpha}$. Because of the aforementioned gauge freedom, the vortex evolution equations are independent of the choice of the open covering of $\mathbf{M}$. What is of physical relevance is not the specific *form* of $\bar{p}_\alpha$ but rather that it is a solution to the partial differential equation (2.25) on *some* open neighborhood U_α containing the point $z^\alpha \in \mathbf{M}$. Such a solution is, of course, by no means unique, and this is the root cause of the above gauge freedom.

To ensure that $\mathcal{K}$ is invariant under uniform rotations and translation, we will henceforth consider the case where n is a constant equal to $1/2\pi$.

$\mathbf{M}$ is a Kähler manifold. As such about *every* point $z_0 \in \mathbf{M}$ there exists an open neighborhood and the choice of a local coordinate system such that $g_{z\bar{z}} = 1 + \tilde{g}_{z\bar{z}}$ where $\tilde{g}_{z\bar{z}}$ vanishes to second order at z_0:

$$\tilde{g}_{z\bar{z}}(z_0) = 0 = \left.\frac{\partial \tilde{g}}{\partial z}\right|_{z_0}. \tag{4.5}$$

Since locally $\mathbf{M}$ behaves very much like the infinite plane (to second order), we expect $\mathcal{L}_v$ to be invariant under uniform rotations and infinitesimal translations (to second order) and, in fact, if we choose the coordinate system given in (3.5), we will of course find the usual conserved charges given in Friedrichs [6]. By doing so, however, we would miss the behaviour of (2.22) under uniform scaling, which is very important in understanding the Kosterlitz–Thouless phase transition. We shall therefore take the time to re-derive the conserved charges from (2.22) by applying (4.5) piecemeal. The usually straightforward procedure of finding the corresponding conserved charges will, however, be complicated by the gauge freedom discussed above.

We start with rotational invariance. Let $w^\alpha = \lambda z^\alpha$ where λ is a complex number. We 'fix' the gauge by requiring that $\bar{p}_\alpha$ be an eigenvector of the rotation operator

$$\bar{p}_\alpha = -\left(z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha}\right)\bar{p}_\alpha \tag{4.6}$$

which is non-singular at $z^\alpha = 0$ (see appendix). Such a $\bar{p}_\alpha$ still satisfies (2.25) and is a *valid choice of gauge*. Requiring that the arc length on $\mathbf{K}$ be invariant:

$$\mathrm{d}s^2 = \sum_\alpha^N q_\alpha \sqrt{g(z^\alpha)}\mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\alpha = \sum_\alpha^N q_\alpha \sqrt{g(w^\alpha)}\mathrm{d}w^\alpha \otimes \mathrm{d}\bar{w}^\alpha \tag{4.7}$$

we obtain

$$\sqrt{g(w^\alpha)} = \frac{\sqrt{g(z^\alpha)}}{|\lambda|^2}\,. \tag{4.8}$$

Defining $\bar{p}'_\alpha(w^\alpha)$ to be the solution to

$$\frac{\partial \bar{p}'_\alpha}{\partial w^\alpha}(w^\alpha) = \sqrt{g(w^\alpha)} \tag{4.9}$$

and choosing the same gauge for $\bar{p}'_\alpha(w^\alpha)$ as for $\bar{p}_\alpha(z^\alpha)$ with $z^\alpha \longrightarrow w^\alpha$ in (4.6), we obtain

$$\bar{p}'_\alpha(w^\alpha) = \frac{\bar{p}(z^\alpha)}{\lambda} \tag{4.10}$$

using (2.25). The Lagrangian then changes by an additive constant

$$\mathcal{L}'_{\rm v}[w^\alpha, \bar{p}'_\alpha] = \mathcal{L}'_{\rm v}[z^\alpha, \bar{p}_\alpha] + \log|\lambda| \sum_{\alpha\neq\beta}^{N} q_\alpha q_\beta \tag{4.11}$$

and the transformation $z^\alpha \longrightarrow \lambda z^\alpha$, $\bar{p}_\alpha \longrightarrow \bar{p}_\alpha/\lambda$ is a symmetry of the system. Letting λ be time-dependent, we find the following conserved charges:

$$I = \tfrac{1}{2}\sum_{\alpha}^{N} q_\alpha (z^\alpha \bar{p}_\alpha + \bar{z}^\alpha p_\alpha) \tag{4.12}$$

which corresponds to pure rotations, and

$$\tilde{I} = \tfrac{1}{2}\mathrm{i}\sum_{\alpha}^{N} q_\alpha (z^\alpha \bar{p}_\alpha - \bar{z}^\alpha p_\alpha) \tag{4.13}$$

which corresponds to pure scaling. Taking the derivatives of both I and $\tilde{I}$ with respect to time, we find that while $\mathrm{d}I/\mathrm{d}t$ requires the use of *both* (4.6) *and* the equations of motion (3.2) to vanish, $\mathrm{d}\tilde{I}/\mathrm{d}t$ vanishes solely due to our choice of gauge. $\tilde{I}$ is therefore *not* dynamical and is instead a 'gauge artefact' which can be set to zero identically (see Chapman [7] for a different approach to scaling).

The charges I and $\tilde{I}$ are conserved as long as (4.6) is a valid gauge choice. This only requires that for every point $z \in \mathbf{M}$ there exists a choice of coordinate system such that $g_{z\bar{z}}$ is rotationally invariant:

$$\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) g_{z\bar{z}} = 0\,. \tag{4.14}$$

This is a somewhat weaker requirement than the Kähler conditions listed in (4.5). A Kähler manifold, of course, has this property.

Considering now infinitesimal translations, let $w^\alpha = z^\alpha + \xi$ where ξ is a complex number. We no longer need require that (4.6) holds. Using the same argument as the above for rotations, $\bar{p}'_\alpha(w^\alpha) = \bar{p}_\alpha(z^\alpha)$ up to a choice of gauge. Then $\mathcal{L}'_{\rm v}[w^\alpha, \bar{p}'_\alpha] = \mathcal{L}_{\rm v}[z^\alpha, \bar{p}_\alpha]$ and the system is invariant under translations (up to a total derivative). The corresponding conserved charges are

$$M = \sum_{\alpha}^{N} q_\alpha p_\alpha \quad \text{and} \quad \overline{M} = \sum_{\alpha}^{N} q_\alpha \bar{p}_\alpha\,. \tag{4.15}$$

We caution, however, that (4.15) is unique only up to a choice of gauge. In fact, using the equations of motion we find that

$$\frac{\mathrm{d}\overline{M}}{\mathrm{d}t} = \sum_{\alpha}^{N} q_\alpha \frac{\partial \bar{p}_\alpha}{\partial z^\alpha} \frac{\mathrm{d}z^\alpha}{\mathrm{d}t}\,. \tag{4.16}$$

Taking the derivative of (2.25) and using (4.5) for $z_0 = z^\alpha$, we find that

$$\frac{\partial}{\partial \bar{z}^\alpha} \frac{\partial \bar{p}_\alpha}{\partial z^\alpha} = 0 \tag{4.17}$$

so that

$$\frac{\partial \bar{p}_\alpha}{\partial z^\alpha} = \frac{\partial h_\alpha}{\partial z^\alpha} \tag{4.18}$$

for some holomorphic function h_α. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{M} - \sum_{\alpha}^{N} q_\alpha h_\alpha\right) = 0 \tag{4.19}$$

and $\overline{M}$ is unique up to a holomorphic function which may be set to zero identically.

If we now require all four conserved charges be conserved simultaneously, then $\bar{p}_\alpha$ must also satisfy (4.6). Setting $\partial \bar{p}_\alpha / \partial z^\alpha = 0$, $\bar{p}_\alpha = \bar{z}^\alpha$ since $g_{z\bar{z}}(z^\alpha) = 1$. With this gauge choice

$$I = \sum_{\alpha}^{N} q_\alpha |z^\alpha|^2 \qquad M = \sum_{\alpha}^{N} q_\alpha z^\alpha \qquad \overline{M} = \sum_{\alpha}^{N} q_\alpha \bar{z}^\alpha \tag{4.20}$$

and $\tilde{I} = 0$. These are the usual expressions for the conserved charges for vortices on the infinite plane and are the most convenient form to use. Note also that when $\mathbf{M}$ is a torus, $g_{z\bar{z}} = 1$ for all $z \in \mathbf{M}$ and $\mathcal{L}_\mathrm{v}$ is invariant under arbitrary *finite* translations as long as the net 'charge' or 'circulation' $Q = \sum_{\alpha}^{N} q_\alpha$ of the vortices vanishes.

Turning our attention now to the generators of rotations and translation, we introduce the following infinite algebra of operators. First, the analogues of the Virasoro algebra for the vortex system

$$T_l \equiv \sum_{\alpha}^{N} (z^\alpha)^{l+1} \frac{\partial}{\partial z^\alpha} \tag{4.21}$$

where l is an integer. Next the functions

$$\begin{aligned} \mathcal{M} &\equiv \sum_{\alpha}^{N} q_\alpha \log |z^\alpha|^2 \qquad & \mathcal{D} &\equiv \sum_{\alpha}^{N} q_\alpha^2 \log |z^\alpha|^2 \\ M_l &\equiv \sum_{\alpha}^{N} q_\alpha (z^\alpha)^l \qquad & D_l &\equiv \sum_{\alpha}^{N} q_\alpha^2 (z^\alpha)^l\,. \end{aligned} \tag{4.22}$$

M_0 is just the net charge Q of the vortices while $M_1 = M$. They have the following commutation relations:

$$\begin{aligned} &[T_l, T_m] = (m-l) T_{l+m} \qquad [T_l, \mathcal{M}] = M_l \qquad [T_l, \mathcal{D}] = D_l \\ &[T_l, M_m] = m M_{m+l} \qquad [T_l, D_m] = m D_{m+l}\,. \end{aligned} \tag{4.23}$$

As will be shown in the next section, the vortex Hamiltonian is simply $\mathcal{K}$. Using the identity

$$(z^\alpha)^{l+1} - (z^\beta)^{l+1} = (z^\alpha - z^\beta)\sum_{k=0}^{l}(z^\alpha)^{l-k}(z^\beta)^k \tag{4.24}$$

we find that for $l \geqslant 0$

$$[T_l, \mathcal{K}] = -\frac{1}{2}\sum_{k=0}^{l} M_{l-k}M_k + \frac{(l+1)}{2}D_l \tag{4.25}$$

while for $l < -1$

$$[T_l, \mathcal{K}] = \frac{1}{2}\sum_{k=-1}^{l+1} M_{l-k}M_k + \frac{(l+1)}{2}D_l \tag{4.26}$$

and $[T_{-1}, \mathcal{K}] = 0$. The infinite set $\mathcal{A} = (T_l, \mathcal{M}, M_l, M_lM_m, \mathcal{D}, D_l, \mathcal{K})$ is then *closed* under the action of T_l. Corresponding to $\mathcal{A}$ there is the complex conjugate set $\overline{\mathcal{A}} = (\overline{T}_l, \overline{\mathcal{M}}, \overline{M}_l, \overline{M}_l\overline{M}_m, \overline{\mathcal{D}}, \overline{D}_l, \overline{\mathcal{K}})$ which is closed under the action of $\overline{T}_l$. As $\mathcal{M}$, D, and K are real functions, they are common to both.

Remarks

(1) If we had kept the self-interaction ($\alpha = \beta$) term in $\mathcal{K}$, there would have been no need to introduce the functions D_l. Indeed, D_l is generated by operating T_l on $\mathcal{D}$, a measure of the system's total self-energy.

(2) T_{-1} is the generator of uniform translations. From (4.25)

$$\tfrac{1}{2}\mathrm{i}\left[(T_0 - \overline{T}_0), \mathcal{K}\right] = 0 \tag{4.27}$$

while

$$\frac{1}{2}\left[(T_0 + \overline{T}_0), \mathcal{K}\right] = -\frac{M_0^2}{2} + \frac{D_0}{2} = -\frac{1}{2}\sum_{\alpha\neq\beta}^{N} q_\alpha q_\beta \tag{4.28}$$

so that linear combinations of T_0 and $\overline{T}_0$ are the generators of rotations and scaling. For uniform translations and rotations we need only consider the *finite*, and *closed* subset $\mathcal{A}_f = (T_{-1}, T_0, M_0, M_1, M_0^2, D_0, \mathcal{K})$ of $\mathcal{A}$.

(3) As is well known, the operators T_{-1}, T_0, and T_1 form a representation of the $su(2)$ Lie algebra. If we add the infinite self-interaction ($\alpha = \beta$) piece to $\mathcal{K}$ and consider the case where M_0, the net vortex charge, is zero, then this new $\mathcal{K}$ is invariant under $su(2)$. Unfortunately to ensure closure the subset of $\mathcal{A}$ that we would have to work with is once again infinite: $\mathcal{A}_{su(2)} = (T_{-1}, T_0, T_1, \mathcal{M}, M_l, \mathcal{K})$.

Returning to the Lagrangian, the conserved charges corresponding to this new symmetry can be found in the usual manner. Let $w^\alpha = \lambda/z^\alpha$. Then $\bar{p}'_\alpha(w^\alpha) = -(z^\alpha)^2\bar{p}_\alpha(z^\alpha)/\lambda$. Using the conditions outlined above, $\mathcal{L}'_v[w^\alpha, \bar{p}'_\alpha] = \mathcal{L}_v[z^\alpha, \bar{p}_\alpha]$ and the transformation $z^\alpha \longrightarrow \lambda/z^\alpha$ is a symmetry of the Lagrangian. Taking λ to be time-dependent, the corresponding conserved charges are once again I and $\bar{I}$. Thus, although $z^\alpha \longrightarrow 1/z_\alpha$ *is* an additional symmetry of the system, no *additional* conserved charges are introduced.

(4) Using (2.25) and the definition of the Kähler potential (3.10), $q_\alpha \bar{p}_\alpha = \partial G/\partial z^\alpha$ up to a gauge choice. With the appropriate choice of gauge, the conserved charges may be expressed in terms of G by

$$\begin{aligned} M &= [\overline{T}_{-1}, G] \qquad \overline{M} = [T_{-1}, G] \\ I &= \frac{1}{2}\left[(T_0 + \overline{T}_0), G\right] \qquad \tilde{I} = \frac{\mathrm{i}}{2}\left[(T_0 - \overline{T}_0)G\right]. \end{aligned} \tag{4.29}$$

Because the Laplace operator commutes with T_0 and $\overline{T}_0$, $g_{z\bar{z}}$ being rotationally invariant means that $\mathrm{i}(T_0 - \overline{T}_0)G/2$ is the sum of a holomorphic and an anti-holomorphic function in the z^αs. This shows explicitly that $\tilde{I}$ is 'pure gauge', and is a description of the underlying structure of **M** itself. The fact that it is conserved simply ensures that $g_{z\bar{z}}$ will always be rotationally invariant.

We end this section with a brief discussion of the discrete symmetries of $\mathcal{L}_\mathrm{v}$, namely parity, time reversal, and charge conjugation. In two dimensions, parity is the map $\mathcal{P} : (x^\alpha, y^\alpha) \longrightarrow (x^\alpha, -y^\alpha)$, or, equivalently, $\mathcal{P} : z^\alpha \longrightarrow \bar{z}^\alpha$. Because $\sqrt{g(z^\alpha)}$ is a real function of z^α, from (2.25) we find that $\mathcal{P} : \bar{p}_\alpha \longrightarrow p_\alpha$. Time reversal $\mathcal{T} : t \longrightarrow -t$. For charge conjugation $\mathcal{C} : q_\alpha \longrightarrow -q_\alpha$, we note that physically the 'charge' of a vortex may be interpreted as the circulation of a singular rotational flow field and thus depends implicitly on the handedness of the local coordinate system that we pick. Moreover, mathematically q_α is identified with the first chern class $c(\mathbf{E})$ of a holomorphic line bundle **E** above **M**. $\mathcal{P} : \mathbf{E} \longrightarrow \overline{\mathbf{E}}$ where $\overline{\mathbf{E}}$ is the complex conjugate bundle, and, as $c(\mathbf{E}) = -c(\overline{\mathbf{E}})$, we conclude that parity is equivalent to charge conjugation for the vortex system. Operating on $\mathcal{L}_\mathrm{v}$ by $\mathcal{P}$ and $\mathcal{T}$, we see that although $\mathcal{L}_\mathrm{v}$ is *not* invariant under the action of either $\mathcal{P}$ or $\mathcal{T}$ separately, it *is* invariant under the combined action of $\mathcal{PT}$. This is the analogeous $\mathcal{CPT}$ operation for the vortex system.

5. Dirac constraints

Up to now we have mostly been concerned with the Lagrangian formulation of the vortex dynamics. As most of the interest in the two-dimensional vortex gas is due to the Kosterlitz–Thouless phase transition, in this section we will attempt to go from Lagrangian dynamics to Hamiltonian dynamics. The form of the vortex Lagrangian that we will be working with is

$$\mathcal{L}_\mathrm{v} = -\mathrm{i}\sum_\alpha^N q_\alpha z^\alpha \frac{\mathrm{d}\bar{z}^\alpha}{\mathrm{d}t} - \mathcal{K} \tag{5.1}$$

where we have set $\bar{p}_\alpha = \bar{z}^\alpha$ and performed an integration by parts. Taking once again $\bar{z}^\alpha$ as our generalized coordinate $\mathbf{q}^\alpha$ with canonical momentum $\mathbf{p}_\alpha = -\mathrm{i}q_\alpha z^\alpha$ we see that the Lagrangian is first-order in the momenta. Consequently, the vortex Hamiltonian $\mathcal{H}_\mathrm{v} \equiv \sum \mathbf{p}_\alpha \dot{\mathbf{q}}^\alpha - \mathcal{L}_\mathrm{v}$ is simply $\mathcal{K}$. More importantly, because of the conserved charges listed in (4.20), not all of the $\mathbf{q}^\alpha$ and $\mathbf{p}_\alpha$ are linearly independent. To address this problem, we use Dirac's theory of contraints [8].

The constraint equations are

$$\phi_z \equiv \sum_\alpha^N \mathrm{i}\mathbf{p}_\alpha - M \qquad \phi_{\bar{z}} \equiv \sum_\alpha^N q_\alpha \mathbf{q}^\alpha - \overline{M} \qquad \phi_0 \equiv \sum_\alpha^N \mathrm{i}\mathbf{q}^\alpha \mathbf{p}_\alpha - I \tag{5.2}$$

which are all *weak* constraints. The vortex Hamiltonian is augmented by a linear combination of these constraints

$$\mathcal{H}_{\mathrm{D}} \equiv \mathcal{H}_{\mathrm{v}} + \sum_{k}^{3} \nu_k \phi_k \tag{5.3}$$

where ν_k are c-numbers which are determined by requiring that ϕ_k be conserved under $\mathcal{H}_{\mathrm{D}}$ as well as $\mathcal{H}_{\mathrm{v}}$. Using the Poisson bracket defined in (3.7), we obtain the following set of linearly dependent equations for ν_k:

$$\nu_{\bar{z}} Q + \nu_0 M = 0 \qquad \nu_z Q + \nu_0 \overline{M} = 0 \qquad \nu_z M - \nu_{\bar{z}} \overline{M} = 0 . \tag{5.4}$$

Defining $\nu_z = \nu \overline{M}$ where ν is a *real* number, we find that $\nu_{\bar{z}} = \nu M$ and $\nu_0 = -\nu Q$. ν is an arbitrary constant which cannot be determined and is due to the invariance of that part of the action corresponding to the free vortex Lagrangian to scaling of t. The Dirac bracket is defined in terms of the Poisson bracket by

$$\{f, g\}_{\mathrm{D}} \equiv \{f, g\} + \mathrm{i}\frac{\left[\{f, \theta_z\}\{\theta_{\bar{z}}, g\} - \{f, \theta_{\bar{z}}\}\{\theta_z, g\}\right]}{|M|^2(1+Q)} \tag{5.5}$$

where

$$\theta_z \equiv \frac{\phi_0}{2} + \overline{M}\phi_z \qquad \theta_{\bar{z}} \equiv \frac{\phi_0}{2} + M\phi_{\bar{z}} \tag{5.6}$$

are chosen so that the Dirac bracket is well defined even when $Q = 0$. The Dirac Hamiltonian is then

$$\mathcal{H}_{\mathrm{D}} = \mathcal{K} + \nu\left(\overline{M}\sum_{\alpha}^{N} q_\alpha z^\alpha + M\sum_{\alpha}^{N} q_\alpha \bar{z}^\alpha - Q\sum_{\alpha}^{N} q_\alpha |z^\alpha|^2\right) - \nu\left(2|M|^2 - QI\right) . \tag{5.7}$$

At this point one may wonder about the usefulness of Dirac's formalism since when analysing the dynamics of a system of vortices one may always put in the constraint (5.2) by hand. This is quite true. If, however, we wish to analyse the statistical-mechanical properties of this system, then the conserved charges of the dynamical system will play a major role. To formulate the grand-canonical ensemble for any dynamical system the constants of the motion are used to construct the Boltzmann factor. To every conserved charge there is a Lagrange multiplier with the term which appears in the Boltzmann factor for the vortex gas having the form $\beta(\mathcal{K} + \bar{\lambda}\sum q_\alpha z_\alpha + \lambda\sum q_\alpha \bar{z}_\alpha + \lambda_{\mathrm{r}}\sum q_\alpha |z_\alpha|^2)$. This is of exactly the same form as the Dirac Hamiltonian (5.7) up to an overall constant. Although the physical interpretations of these Lagrange multipliers are straightforward, λ and $\bar{\lambda}$ being the componants of an external flow field while λ_{r} is the vorticity of a flow field, the relative signs are not determined using the simple Lagrange multipier prescription, as it is using the Dirac Hamiltonian. Moreover, while λ_{r} can be interpreted as a net vorticity, from (5.7) we see that this net vorticity must also include the internal vorticity of the system, namely that due to the vortices themselves. As such, even in the absence of an external flow field, a system with an odd number of vortices, all of charge ± 1, will have $Q \neq 0$. An additional term now appears in the Boltzmann factor which will change the statistics of the system dramatically.

The Dirac formulation thus gives a systematic way of constructing the Boltzmann factor for the vortex system. This is to be contrasted with the method that Kosterlitz and Thouless first used in [9] to construct their ensemble. A uniform 'electric field' $\boldsymbol{E}$ was introduced by hand and coupled linearly to the vortices by adding the term $\sum q_\alpha \boldsymbol{E}\cdot\boldsymbol{r}_\alpha$ to the coulombic interaction Hamiltonian. This was done so as to define a 'polarizability' for the system in the dipole phase of the phase transition which subsequently lead to one of the renormalization group equations. By defining $E_x = \left(M+\overline{M}\right)/\sqrt{2}$, $E_y = \mathrm{i}\left(\overline{M}-M\right)/\sqrt{2}$, we see that this term that Kosterlitz and Thouless put in using physical arguments arises quite naturally from the Dirac Hamiltonian. Moreover, up until recently there has not been a systematic derivation of the superfluid density that Nelson and Kosterlitz [10] used in deriving the celebrated universal jump in superfluid density in two dimensions. This was done by Minnhagen and Warren [11] by using a linear response theory which coupled the motion of the vortices to a uniform flow field and it also arises from that part of $\mathcal{H}_\mathrm{D}$ that is linear in z^α. Taking each U_α to be a circle now centred about the origin

$$z^\alpha = \frac{1}{2\pi\mathrm{i}}\int_{U_\alpha}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\bar{z}-\bar{z}^\alpha} \tag{5.8}$$

exactly. Defining

$$v^\alpha_{sf} \equiv \frac{q_\alpha}{2\pi\mathrm{i}}\left(\frac{\mathrm{d}z}{z-z^\alpha}-\frac{\mathrm{d}\bar{z}}{\bar{z}-\bar{z}^\alpha}\right) \tag{5.9}$$

and $v_\mathrm{ext} = -\mathrm{i}M\mathrm{d}z + \mathrm{i}\overline{M}\mathrm{d}\bar{z}$, we obtain

$$\sum_\alpha^N \overline{M}z^\alpha + M\bar{z}^\alpha = \int_\mathrm{M} \boldsymbol{v}_\mathrm{ext}\cdot\boldsymbol{v}_\mathrm{sf}\,\mathrm{d}^2x \tag{5.10}$$

where $\boldsymbol{v}_\mathrm{sf} = \sum \boldsymbol{v}^\alpha_\mathrm{sf}$. In rectilinear coordinates,

$$\boldsymbol{v}^\alpha_\mathrm{sf} = \frac{q_\alpha}{2\pi\mathrm{i}}\left[-\frac{(y-y^\alpha)}{|\boldsymbol{r}-\boldsymbol{r}^\alpha|^2}\hat{e}_x + \frac{(x-x^\alpha)}{|\boldsymbol{r}-\boldsymbol{r}^\alpha|^2}\hat{e}_y\right] \tag{5.11}$$

and was interpreted by Minnhagen and Wagner as the superfluid flow field from a vortex with charge q_α. Then $\boldsymbol{v}_\mathrm{sf}$ is the net superfliud flow and was labelled by them as $\boldsymbol{g}_\perp$. $\boldsymbol{v}_\mathrm{ext}$ is identified with the uniform flow they introduced.

These additional pieces to the Boltzmann factor, which is not present in the standard renormalization group analysis of the phase transition, play a great role determining the properties of the system away from criticality. By using the Dirac Hamiltonian for $\nu = -1$ for the general Kosterlitz–Thouless ensemble (in which the charges of the vortices can be any integer) it is shown in [12] that many of the properties of the system that were believed to be true about the phase transition actually were not. In particular, when $Q = 0$ we find that

$$\langle\delta H\rangle|_{M=0} \neq \lim_{M\to 0}\langle\delta H\rangle \tag{5.12}$$

where $\delta H = -(M\sum q_\alpha\bar{z}^\alpha + \bar{M}\sum q_\alpha z^\alpha)$ and the average was calculated using the grand partition function

$$\mathcal{Z} = \sum_\mathrm{config} g(N,q)\lambda(N,q)Z_N[M] \tag{5.13}$$

and

$$Z_N[M] = \prod_{\alpha}^{N} \int \frac{\mathrm{i}dz^\alpha \wedge d\bar{z}^\alpha}{a^2} \exp\left(-\beta \left[\mathcal{K} - (\bar{M} \sum q_\alpha z^\alpha + M \sum q_\alpha \bar{z}^\alpha)\right]\right) . \tag{5.14}$$

a is the length scale for the system while $\lambda(N, q)$ and $g(N, q)$ are the fugacity and multiplicity factors due to identical particles respectively. (Our notation and units here differ somewhat from the notation used in [12]). By $\langle \delta H \rangle|_{M=0}$ we mean the average of δH where $M = 0$ identically. This is of course zero. The presence of δH in (5.14) when $M \neq 0$ breaks rotational symmetry of the vortices, $z^\alpha \to e^{i\theta} z^\alpha$, and the non-equality in (5.12) mean that the phase transition breaks a continuous rotational symmetry. This is in direct contrast to the Hohenberg–Mermin–Wagner theorem which states that there is no phase transition in two dimensions which breaks a continuous symmetry [13, 14]. Moreover, it may also be shown that for the system to behave as a normal fluid above the transition temperature and a superfluid below it, there can be no vortices above the transition temperature, while there must be vortices below it. This is also in direct contrast with what is currently believed about the Korterlitz–Thouless phase transition.

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Appendix

What appears in the vortex evolution equations, and what is of physical relevance, is the metric $g_{z\bar{z}}$ evaluated at the point $z^\alpha \in \mathbf{M}$, not $\bar{p}_\alpha$. Unfortunately, what appears in $\mathcal{L}_v$ is $\bar{p}_\alpha$, not $g_{z\bar{z}}$. Although $\bar{p}_\alpha$ *is* related to $g_{z\bar{z}}$ through (2.25), this is true for *any* metric $g_{z\bar{z}}$ of $\mathbf{M}$ irrespective of any symmetries that the manifold may have. Suppose now that for every $z \in \mathbf{M}$ there exists a choice of local coordinate about z such that (4.14) holds. For $\mathcal{L}_v$ to be rotationally invariant, a specific $\bar{p}_\alpha$ must be found which reflects this symmetry. As we shall see, the most natural choice is (4.6). We start by outlining the following facts about the rotation operator.

Fact 1. Let E be a C^∞ function of $z, \bar{z}$ defined everywhere an open neighborhood U of a Riemann Surface $\mathbf{M}$ except, perhaps, at $z = 0$. Then E is an eigenvector of the rotation operator

$$\mathcal{R} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} \tag{A1}$$

with eigenvalue e, a complex number, iff

$$E = \left(\frac{z}{\bar{z}}\right)^{e/2} f(|z|) \tag{A2}$$

where f is any C^∞ function of $|z|$ only.

Proof. Suppose that E is a eigenvector of $\mathcal{R}$ with eigenvalue e:

$$eE = \mathcal{R}E\,. \tag{A3}$$

Let $\chi = |z|$, $\xi = z/\bar{z}$. Then

$$eE = 2\xi\frac{\partial E}{\partial \xi} \tag{A4}$$

and $E = \xi^{e/2}f(\chi)$ where f is any C^∞ function of χ only. The converse follows straightforwardly.

Fact 2. If E is an eigenvector of $\mathcal{R}$ with eigenvalue e iff $\overline{E}$ is an eigenvector of $\mathcal{R}$ with eigenvalue $-\bar{e}$.

Fact 3. If E is an eigenvector of $\mathcal{R}$ with eigenvalue e, then

$$\frac{\partial^{l+m}E}{\partial z^l \partial \bar{z}^m} \tag{A5}$$

is also an eigenvector of $\mathcal{R}$ with eigenvalue $(e - l + m)$.

Proof. By induction. It is certainly true for $l, m = 0$. Assume that it is true for $l = r$ and $m = s$. Then

$$(e - r + s)\frac{\partial^{r+1+s}E}{\partial z^{r+1}\partial \bar{z}^s} = \frac{\partial}{\partial z}\left(\mathcal{R}\frac{\partial^{r+s}E}{\partial z^r \partial \bar{z}^s}\right)\,. \tag{A6}$$

Since

$$\left[\frac{\partial}{\partial z}, \mathcal{R}\right] = \frac{\partial}{\partial z} \tag{A7}$$

$$(e - r + s)\frac{\partial^{r+1+s}E}{\partial z^{r+1}\partial \bar{z}^s} = \frac{\partial^{r+1+s}E}{\partial z^{r+1}\partial \bar{z}^s} + \mathcal{R}\frac{\partial^{r+1+s}E}{\partial z^{r+1}\partial \bar{z}^s} \tag{A8}$$

and the assertion is true for the first index l. By the same argument, it also holds for the second index m, and we are done.

The converse is not true in general. Take, as the simplest example, the case where $\partial E/\partial z = 1$ which is an eigenvector of $\mathcal{R}$ with eigenvalue 0. Then $E = z + \bar{h}(\bar{z})$, where $\bar{h}$ is any antiholomorphic function. E is an eigenvector of $\mathcal{R}$ with eigenvalue 1 iff $h \equiv 0$. The degree to which it 'fails' to be an eigenvector is the additional arbitrary antiholomorphic function.

$g_{z\bar{z}}$ is an eigenvector $\mathcal{R}$ with eigenvalue 0. For $\bar{p}_\alpha$ to reflect this symmetry, we *require* that $\bar{p}_\alpha$ be an eigenvector of $\mathcal{R}$ with eigenvalue -1. Such a $\bar{p}_\alpha$ satisfies (2.25) by construction. From (A2) and (A5),

$$\bar{p}_\alpha = \frac{1}{z_\alpha}\int_0^{|z^\alpha|^2} g_{z\bar{z}}(r)\mathrm{d}r + \frac{k}{z^\alpha} \tag{A9}$$

where k is any complex number. Requiring that $\bar{p}_\alpha$ be non-singular at $z^\alpha = 0$ *uniquely* defines $\bar{p}_\alpha$ in terms of $g_{z\bar{z}}$ and $\bar{p}_\alpha$ is completely determined by our gauge choice.

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